



# Non-local rheology in granular media: a perspective on the 2015 EPJE Paper by Bouzid et al.

Olivier Pouliquen<sup>1,a</sup>

<sup>1</sup> IUSTI, CNRS, Aix Marseille Univ, Marseille 13013, France

Received 10 February 2026 / Accepted 19 March 2026

© The Author(s), under exclusive licence to EDP Sciences, SIF and Springer-Verlag GmbH Germany, part of Springer Nature 2026

**Abstract** The study "Non-local rheology in dense granular flows: Revisiting the concept of fluidity," published in 2015 in The European Physical Journal E (vol. 38) by Mehdi Bouzid and collaborators, stands as an important contribution to the rheology of granular materials. In their work, the authors critically discuss the differences between proposed non-local models and provide clear pathways to discriminate between them. This perspective paper revisits the state of the art at the time of the Bouzid et al's publication, highlighting its role in inspiring subsequent research. We then explore recent advancements since 2015, which, while significant, have not yet fully resolve the questions originally raised by Bouzid et al.

## 1 Preface

Granular media, beyond their importance in numerous industrial applications and in describing geophysical phenomena such as avalanches or landslides, provide a stimulating playground for physicists. The behavior of a large collection of non-Brownian rigid grains reveals remarkable complexity and poses many challenges at the interface of statistical physics, fluid and solid mechanics, rheology, and soft matter. One of the major questions in this field is whether a continuous description of these particulate materials is possible. Can we propose constitutive laws that would provide the foundation for a granular hydrodynamics, thereby enabling quantitative predictions of flows in silos, avalanches, and other systems? In the 2000 s, significant efforts were made to understand and describe granular media as a continuous fluid [1], leading to the development of relatively simple rheological models. However, it quickly became apparent that, despite their evident successes in capturing many phenomena, these approaches had limitations, particularly in accounting for finite-size effects and non-local phenomena. The 2015 paper by Bouzid et al. in EPJE [2] is emblematic of this research, highlighting the importance of non-local effects in granular rheology and offering a comparison of various models proposed at the time.

On the occasion of the 25th anniversary of the European Physical Journal E, I propose in this perspective a comment of this influential paper. I will first outline the state of the art and open questions prior to its publication, then provide a concise summary of the ideas

and results presented in the paper, and finally discuss how these questions have evolved and progressed over the past decade, and what new challenges now face the community.

## 2 State-of-the-art prior Bouzid et al (2015)

### 2.1 The local rheology

The two decades from 1990 to 2010 saw major advances in the description and understanding of granular flows. Earlier efforts had primarily focused on characterizing the very slow deformation regimes typically encountered in soil mechanics, leading to the development of plasticity models for granular materials [3]. The question of how to describe granular flows really started with the development of kinetic theory for granular media [4,5]. The idea was to treat flowing grains as a gas of hard, inelastic particles undergoing binary collisions. By introducing inelasticity and energy dissipation into the kinetic theory equations for gases, a set of constitutive laws (for mass, momentum, and energy) was derived, based on density, velocity, and granular temperature (defined as velocity fluctuations). While this approach successfully described dilute and highly agitated granular media, its early versions rooted in collisional dynamics, failed to capture dense flows and the scaling laws observed in inclined plane flows, vertical chute flows, and rotating drums.

A phenomenological approach thus emerged, partly inspired by collective work from the GDR Midi group [1], which compiled experimental and simulation data from multiple studies. This effort highlighted an extre-

<sup>a</sup> e-mail: [olivier.pouliquen@univ-amu.fr](mailto:olivier.pouliquen@univ-amu.fr) (corresponding author)

mely fruitful dimensional analysis when considering the pressure-imposed rheology of a granular medium made of rigid particles. When rigid particles of size  $d$  and density  $\rho_s$  are sheared at a shear rate  $\dot{\gamma}$  under a confining pressure  $P$ , dimensional analysis dictates that the shear stress  $\tau$  and the volume fraction  $\phi$  are controlled by a single dimensionless number: the inertial number  $I = \dot{\gamma}d/\sqrt{P/\rho_s}$  [1, 6]. This leads to a frictional law and a dilatancy law:

$$\tau = \mu(I)P, \quad \phi = \phi(I) \quad (1)$$

The frictional law  $\mu(I)$  is an increasing function of  $I$ , starting from a critical threshold value  $\mu_c$  as  $I \rightarrow 0$  (quasi-static regime), while  $\phi(I)$  is a decreasing function of  $I$ , starting from the jamming volume fraction  $\phi_c$  as  $I \rightarrow 0$ . This local description, where the stress at a point depends solely on the local shear rate via  $\tau = \mu(I)P$ , was later generalized into a tensorial formulation [7], successfully capturing many observations across different flow configurations [8–10]. However, limitations of this local rheology soon became apparent.

Several observations highlight the limitations of a purely local rheology, particularly in slow flows near the quasi-static regime. A first signature is found in the velocity profiles observed in various flow configurations, such as surface flows in rotating drums, Couette flows, and planar shear under gravity [1]. In all these setups, the stress distribution leads to a friction coefficient that exceeds the threshold  $\mu_c$  only in a first zone, falling below it in a second zone. Local rheology would predict no shear in the sub-threshold zone. However, a systematic "creep" flow characterized by exponential velocity profiles is consistently observed. This suggests that the sub-threshold zone receives fluctuations from the zone above the threshold, enabling it to flow: a non-local phenomenon.

A second signature of non-locality is observed in finite-size effects. The critical angle  $\theta$ , at which a layer of grains is able to flow, depends on the layer thickness  $h$  and increases when  $h$  becomes smaller than a few tens of grain diameters [11]. A local rheology would predict a single limiting angle regardless of thickness.

A final set of experiments shows that the behavior of a granular medium near a sheared zone can be profoundly altered. For example, a steel ball placed at the surface of a granular packing subjected to a basal shear below the surface, begins to flow, even though no surface flow is present [12]. This suggests that the presence of deep shear weakens the medium beneath the ball, causing it to sink. A variant of this experiment involved moving a rod near a shear band in a region supposed to be below the flow threshold, demonstrating that the rod's velocity appears to follow an Arrhenius-like law, with an energy barrier corresponding to the deviation from the spontaneous motion threshold [13]. All these observations point to intrinsic non-local effects linked to the discrete nature of the medium. The key ques-

tion remained: Could these effects be incorporated into a continuous description?

## 2.2 Different non-local models

To incorporate non-local effects into a continuous model, several approaches have been proposed. Some rely on an integral formulation to capture the influence of neighboring regions, inspired from Eyring theory [14] or by introducing averaging kernels in plasticity models [15]. Other approaches focus on the role of velocity fluctuations, building on modified kinetic theory [16] or generalized energy fluctuation equations [17].

At the time of Bouzid et al.'s paper, two approaches were gaining significant attention within the community. Both introduced diffusive terms into phenomenological constitutive equations to account for non-locality [18, 19]. Although the two approaches share common formalisms, they also exhibit important differences, which lie at the heart of the discussion in Bouzid et al.'s paper [2]. The first approach, developed by Kamrin and collaborators in a series of papers [18, 20, 21] is inspired by Kinetic Elasto-Plastic (KEP) theory, originally formulated for elasto-plastic fluids such as foams and emulsions [22, 23]. This theory derives a non-local description from the statistics of localized plastic events, introducing the concept of fluidity  $f$ , defined as the inverse of viscosity:  $f = \dot{\gamma}/\tau$ , where  $\dot{\gamma}$  is the shear rate and  $\tau$  is the shear stress. KEP theory proposes an evolution equation for  $f$  that includes a diffusive term, accounting for the spatial range of elastic relaxation during a localized plastic event.

To adapt this theory to granular media, Kamrin et al. [18, 20] proposed replacing the stress  $\tau$  with the friction coefficient  $\mu$  in the definition of a granular fluidity, denoted as  $g$ :

$$g = \dot{\gamma}/\mu \quad (2)$$

The fluidity equation then takes the form:

$$g = g_{loc} + \xi^2 \Delta g, \quad (3)$$

where  $g_{loc}(\mu, P)$  is the local fluidity, which is zero if  $\mu < \mu_c$  and otherwise given by:  $g_{loc}(\mu, P) = I(\mu)/(T_i\mu)$ , with  $I(\mu)$  corresponding to the local rheology (inverse of Eq. 1.) and  $T_i = d/\sqrt{P/\rho_s}$  representing the inertial time scale. The characteristic length scale of non-local effects,  $\xi$ , is assumed to diverge symmetrically on either side of the friction coefficient threshold  $\mu_c$ :

$$\frac{\xi}{d} = A \sqrt{\frac{1}{|\mu - \mu_c|}}. \quad (4)$$

with  $A$  a phenomenological constant.

The second approach, proposed by Bouzid et al. [19], considers the inertial number  $I$  itself as a state variable that diffuses due to non-local effects, such as correlated motions or soft modes observed near jamming transitions. Bouzid et al. introduce a non-local rheology by expressing the friction coefficient as its local value,

reduced by a term related to the diffusion of the inertial number:

$$\mu = \mu_{loc}(I) \left( 1 - \nu d^2 \frac{\Delta I}{I} \right), \quad (5)$$

where  $\nu$  is a constant. Here, the typical diffusion length of the state parameter is  $\nu d^2/I$ , which diverges as  $I \rightarrow 0$ . The local friction coefficient  $\mu_{loc}(I)$  is given by the standard local rheology Eq. 1.

While these two approaches share similarities in their formalism, they also exhibit fundamental differences that form the core of the discussion in Bouzid et al.'s paper [2]. They offer complementary perspectives for capturing non-local effects but differ in their choice of state variable. This has significant implications for predicting flow behavior, particularly near the jamming transition.

### 3 The main messages of Bouzid et al (2015)

The paper by Mehdi Bouzid and collaborators was published at a time when several approaches to non-local rheology for granular materials had already been proposed. In this important paper, the authors aim to clarify the difference between the approaches by discussing the foundations of each model, focusing specifically on the fluidity model [20] (Eq. 3) and the gradient expansion model [19] (Eq. 5). Their goal was to assess the relevance and predictive power of these models across various flow configurations. Below, we summarize the paper's key findings.

#### 3.1 The basic foundation of the fluidity

The non-local granular fluidity model draws inspiration from the elasto-plastic non-local theory proposed by Bocquet et al [22]. While the latter was derived by considering the statistics and dynamics of elastic loading/plastic event cycles in elasto-plastic fluids, Bouzid et al. emphasize that its generalization to granular media is far from straightforward. Granular systems operate in a regime where deformation phenomenology differs significantly, governed by strong correlations (soft modes) rather than localized plastic event avalanches. To highlight this difference, Bouzid et al. compare discrete simulations of simple shear for soft and rigid grains. They show that the deformation dynamics are markedly distinct: soft systems deform through successive phases of elastic loading followed by irreversible plastic event avalanches, whereas rigid systems exhibit much more frequent and spatially distributed rearrangements. This observation suggests caution when interpreting granular fluidity in comparison to the fluidity of elasto-plastic media.

#### 3.2 Relaxation length in inhomogeneous flows

To further compare the models, Bouzid et al. discuss the predicted velocity profiles in flows where stress distribution varies. By varying the pressure near boundaries, they force the flow to transition from above the yield threshold to a lower value, either above or below the friction threshold. The junction between these two zones exhibits a relaxation of the shear rate over a measurable length. Their data show that this length diverges as the friction coefficient approaches its threshold value, both from above and below. A slight asymmetry in the divergence is observed, more pronounced in frictionless particles. Both the fluidity and gradient expansion non-local models predict this divergence in the relaxation length of  $\dot{\gamma}$  as  $\mu \rightarrow \mu_c$ , although the observed asymmetry is only predicted by Bouzid et al.'s model. The authors conclude that comparing velocity profile predictions alone is insufficient to discriminate between the models, as both predict a similar divergence in relaxation lengths near the flow threshold.

#### 3.3 A microrheometer to investigate non-local effects

To explore this further, the authors propose an ingenious configuration [24]: locally probing a flow driven by boundaries with an additional small shear. The technique involves imposing a tangential stress jump at a given depth in a planar shear flow by applying a horizontal force only to particles crossing a line. In this configuration, which features a stress discontinuity, the fluidity and gradient expansion models predict different jump conditions. In the fluidity model,  $g = |\dot{\gamma}|/\mu$  is assumed to be continuous. Since  $\mu$  is discontinuous,  $|\dot{\gamma}|$  must also be discontinuous. In contrast, the gradient expansion model assumes that the inertial number  $I$  is continuous, implying that  $|\dot{\gamma}|$  must remain continuous. Discrete simulations support this hypothesis, showing that the ratio of shear rates above and below the stress jump remains equal to 1, independent of the stress jump magnitude. Although this result pertains to a specific configuration, it appears strong enough to discriminate between the models. In my opinion, its significance has been largely underestimated by the community.

#### 3.4 The main conclusions of Bouzid et al (2015)

The main conclusion of Bouzid et al.'s paper is that the fluidity-based non-local model and the gradient expansion non-local model, while sharing similarities in their formulation, are not equivalent. Although both models provide comparable predictions for velocity profiles in classical inhomogeneous flows, such as Couette flows, they are fundamentally different in their underlying conceptualization of non-locality. The fluidity model is rooted in the idea of localized plastic events, whereas the gradient expansion model is based on correlated soft modes. Moreover, the two models do not predict the same behavior in configurations involving stress dis-

continuities. The authors emphasize the need for further studies to analyze the mechanisms underlying non-locality, a question that has inspired significant research since the publication of their paper.

## 4 What's new since Bouzid et al (2015)

Since the publication of Bouzid et al.'s paper, the question of non-locality in granular rheology has remained a dynamic and active field of research [25]. Several key developments have emerged. First, following Bouzid et al.'s conclusion remarks, many studies have focused on identifying the relevant underlying variables that might explain non-local effects. A second, indirectly related line of investigation concerns the well-posedness of constitutive equations [26]. The widely used  $\mu(I)$  rheology has been shown to be ill-posed, exhibiting inherent instabilities at small wavelengths, a major challenge for applications in numerical simulations. Introducing non-local terms in the rheology can be seen as a physical means of regularization, though this approach has proved to be far from straightforward. The relationship between non-locality and the well-posedness of constitutive equations has thus become a critical area of study. Below, I subjectively highlight several examples of recent developments in this evolving field.

### 4.1 Origin of non-locality

Following the relative success of the fluidity model in predicting several flow features in numerical and experimental studies, many researchers have sought to interpret fluidity in terms of microscopic dynamics [27–34]. A popular candidate for the field variable underlying fluidity is the mean velocity fluctuations, which carry information about both local agitation and spatial correlations. Several numerical studies have simulated different flow configurations, particularly inhomogeneous flows where non-local creep velocity profiles are present [27, 29, 30]. These studies computed the fluidity  $\dot{\gamma}/\mu$  directly from local measurements of the shear rate and stresses, while also measuring the locally averaged velocity fluctuations  $\overline{\delta v}$ . A master curve emerges when plotting the dimensionless fluidity  $gd/\overline{\delta v}$  as a function of the local volume fraction  $\phi$  [27], though the quality of this collapse may depend on how the fluctuations are averaged [29], or if the shear is not unidirectional [30].

This correlation suggests a direct relationship between the friction coefficient and granular temperature, without explicitly introducing the volume fraction. Kim and Kamrin [35], simulated various flow configurations with creep regions below the flow threshold, computing the friction coefficient  $\mu$ , the inertial number  $I$ , and the dimensionless granular temperature  $\Theta = \rho \overline{\delta v}^2 / P$  along the flow profile. While a good collapse of  $\mu(I)$  and  $\Theta(I)$  is observed in the inertial regime, where local rheology is expected to hold, this relationship breaks down when  $I$  falls below 0.05, in the quasi-static regime. Here,

no one-to-one relationship exists between  $\mu$  and  $I$ , or between  $\Theta$  and  $I$ . However, a remarkable collapse is observed when plotting  $\mu\Theta^{1/6}$  as a function of  $I$  across the entire range of  $I$  [32, 35]. This strongly suggests the existence of a universal function  $\mu(I, \Theta)$  valid everywhere. This scaling has been also tested experimentally [32, 33]. Interestingly, a recent study shows that even in immersed granular media, where dissipation occurs through the interstitial fluid, a similar scaling is observed [36]. However, this scaling alone does not constitute a constitutive law, as it still requires proposing an evolution equation for the granular temperature  $\Theta$ . Interestingly, if one expands  $\Theta$  in terms of gradients of  $I$ , the gradient expansion proposed by Bouzid et al. [19] might be recovered.

At this stage, if granular temperature emerges as a relevant field, it is legitimate to ask whether kinetic theory of granular media could provide a complete framework, as it already offers hydrodynamic equations for volume fraction, velocity, and granular temperature. Berzi [16] demonstrated how a connection can indeed be made between kinetic theory and the non-local fluidity model, where non-locality arises from the conduction of granular temperature in the energy equation. However, it is important to note that for kinetic theory to capture the dense flowing regime and the local rheology, several phenomenological closures must be introduced, particularly for a cooperative length scale. This makes the theory as phenomenological as the other approaches, and while the structure of kinetic theory equations may be relevant, its interpretation in terms of collisional dynamics is questioned. For instance, the observation that the same temperature scaling is observed in overdamped systems in suspension [36], suggests that fluctuations are primarily controlled by geometrical deformation modes, as proposed for example by Sheraki et al [37], and not by inelastic collisions. The role of fluctuations is also emphasized in a recent theoretical study [17], demonstrating how a perturbative analysis of a general energy equation, not limited to collisional dynamics, can capture the scaling relationship between granular temperature and the friction coefficient. This approach successfully recovers a non-local fluidity model similar to those previously proposed.

Overall, despite the observation of intriguing scalings, the closure of non-local models and the link with microscopic dynamics remain incomplete. The field continues to explore how these insights can be integrated into a robust theoretical framework.

### 4.2 Link between ill-posedness and non-local models

The motivation to develop non-local models in the 2010s was initially driven by the limitations of the local  $\mu(I)$  rheology, particularly in capturing velocity profiles and the long-distance influence of shear. However, another critical limitation of the local rheology emerged after the publication of Bouzid et al.'s work: the ill-posedness of the incompressible local rheology. Through stability analysis of plane shear flows, Baxter

et al. [38] demonstrated that the system is unstable to short-wavelength perturbations, with diverging growth rates, when the inertial number  $I$  is either very low or very high. This ill-posedness manifests, for example, in numerical simulations where spurious unstable waves appear, whose properties are dependent on the mesh size. The origin of this instability can be attributed to a non-convex energy potential, and studies [39, 40] have discussed how a proper theoretical framework based on thermomechanics considerations might help in formulating well-posed constitutive laws.

Several approaches have been proposed to address this ill-posedness. One approach involves modifying the functional form of the  $\mu(I)$  rheology to eliminate the instability, which requires regularization at both low and high inertial numbers [41, 42]. Another approach abandons the incompressibility assumption and accounts for variations in the volume fraction [43–45]. Concerning the non-local models, at first glance, one might think that adding a diffusive Laplacian term to the constitutive laws would regularize and stabilize small wavelengths and provide a well-posed rheology. However, as shown by Li and Henann [46], the situation is more complex. While the fluidity model appears to be well-posed, meaning that plane shear is unconditionally stable, the gradient expansion constitutive law as initially proposed by Bouzid et al. [19] is actually even more ill-posed than the simple local rheology. The source of this instability seems to be the presence of pressure in the Laplacian term through the inertial number  $I$ . Naïvely, one might wonder whether this weakness could be mitigated by using the norm of the shear rate  $|\dot{\gamma}|$  instead of the inertial number  $I$  as the state parameter.

## 5 Conclusion

The paper by Bouzid et al., published in EPJE in 2015, which compared non-local approaches, raised several important points to evaluate and understand the differences between the models. First, the authors emphasized the distinction between the classical elasto-plastic scenario, from which the concept of fluidity was originally derived, and the case of rigid granular media, where non-locality arises more from correlated soft modes rather than localized plastic events. A second, equally important result from Bouzid et al., which may have been underestimated in some recent studies, is the necessity to test non-local models not only by comparing velocity profiles but also by examining configurations that locally probe the rheology.

While the question of a well-posed, physically grounded non-local rheology remains open [25], both the fluidity approach and the gradient expansion model have been adopted in several numerical studies [47–51], despite the questions about their microscopic origins, relative merits, and the ill-posedness of the gradient expansion model. These studies have successfully captured non-trivial non-local effects and provided more realistic predictions of velocity profiles across various

configurations compared to local rheology. Experiments have also been conducted to test these predictions and link macroscopic fields with microscopic fluctuations [28, 32–34]. Additionally, several enrichments have been proposed, including the incorporation of hysteresis [52] and normal stress differences [53] within the fluidity framework.

Open questions still remain. The question of boundary conditions is not yet very clear. The importance of non-locality in cohesive granular media is another active area of research [54], as intrinsic localization phenomena appear to be a generic feature of these materials. Their description within a continuum approach may require even more consideration of non-local effects for regularization than in dry granular media, as demonstrated in studies where the Bouzid et al. model successfully predicts the characteristics of shear banding in cohesive granular media [55]. There is no doubt that non-locality will continue to inspire and motivate future studies.

**Acknowledgements** The author acknowledges the European Research Council (ERC) under the European Union's Horizon (Grant 101097842), financial support from International Fine Particle Research Institute (IFPRI) and the Agence Nationale de la Recherche under RheoCom (ANR-22-CE06-0020) grant.

## References

1. GDR-MiDi. On dense granular flows, *Eur. Phys. J. E* **14**, 341–365 (2004)
2. M. Bouzid, A. Izzet, M. Trulsson, E. Clément, P. Claudin, B. Andreotti, Non-local rheology in dense granular flows: Revisiting the concept of fluidity. *Eur. Phys. J. E* **38**, 1–15 (2015)
3. A.N. Schofield, P. Wroth, *Critical state soil mechanics* (McGraw-hill, London, 1968)
4. J.T. Jenkins, S.B. Savage, A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles. *J. Fluid Mech.* **130**, 187–202 (1983)
5. I. Goldhirsch, Rapid granular flows. *Ann. Rev. Fluid Mech.* **35**(1), 267–293 (2003)
6. F. Da Cruz, S. Emam, M. Prochnow, J.-N. Roux, F. Chevoir, Rheophysics of dense granular materials: discrete simulation of plane shear flows. *Phys. Rev. E* **72**(2), 021309 (2005)
7. P. Jop, Y. Forterre, O. Pouliquen, A constitutive relation for dense granular flows. *Nature* **44**, 727–730 (2006)
8. P.-Y. Lagrée, L. Staron, S. Popinet, The granular column collapse as a continuum: validity of a two-dimensional navier-stokes model with a  $\mu(I)$ -rheology. *J. Fluid Mech.* **686**, 378–408 (2011)
9. L. Staron, P.-Y. Lagrée, S. Popinet, Continuum simulation of the discharge of the granular silo: a validation test for the  $\mu$  (i) visco-plastic flow law. *Eur. Phys. J. E* **37**(1), 5 (2014)

10. S. Dunatunga, K. Kamrin, Continuum modelling and simulation of granular flows through their many phases. *J. Fluid Mech.* **779**, 483–513 (2015)
11. O. Pouliquen, Scaling laws in granular flows down rough inclined planes. *Phys. Fluids* **11**(3), 542–548 (1999)
12. K. Nichol, A. Zanin, R. Bastien, E. Wandersman, M. Hecke, Flow-induced agitations create a granular fluid. *Phys. Rev. Lett.* **104**, 078302 (2010)
13. K.A. Reddy, Y. Forterre, O. Pouliquen, Evidence of mechanically activated processes in slow granular flows. *Phys. Rev. Lett.* **106**, 108301 (2011)
14. O. Pouliquen, Y. Forterre, A non-local rheology for dense granular flows. *Phil. Trans. R. Soc. A* **367**(1909), 5091–5107 (2009)
15. P.V. Dsouza, P.R. Nott, A non-local constitutive model for slow granular flow that incorporates dilatancy. *J. Fluid Mech.* **888**, 3 (2020)
16. D. Berzi, On granular flows: from kinetic theory to inertial rheology and nonlocal constitutive models. *Phys. Rev. Fluids* **9**(3), 034304 (2024)
17. B.M. Alessio, M.R. Edwards, C.Y. Lai, Dense granular rheology from fluctuations. *arXiv* **2601**, 01907 (2026)
18. D.L. Henann, K. Kamrin, A predictive, size-dependent continuum model for dense granular flows. *Proc. Natl. Acad. Sci.* **110**(17), 6730–6735 (2013)
19. M. Bouzid, M. Trulsson, P. Claudin, E. Clément, B. Andreotti, Nonlocal rheology of granular flows across yield conditions. *Phys. Rev. Lett.* **111**(23), 238301 (2013)
20. K. Kamrin, G. Koval, Nonlocal constitutive relation for steady granular flow. *Phys. Rev. Lett.* **108**(17), 178301 (2012)
21. K. Kamrin, D.L. Henann, Nonlocal modeling of granular flows down inclines. *Soft Matter* **11**(1), 179–185 (2015)
22. L. Bocquet, A. Colin, A. Ajdari, Kinetic theory of plastic flow in soft glassy materials. *Phys. Rev. Lett.* **103**(3), 036001 (2009)
23. A. Nicolas, E.E. Ferrero, K. Martens, J.-L. Barrat, Deformation and flow of amorphous solids: Insights from elastoplastic models. *Rev. Mod. Phys.* **90**(4), 045006 (2018)
24. M. Bouzid, M. Trulsson, P. Claudin, E. Clément, B. Andreotti, Microrheology to probe non-local effects in dense granular flows. *Europhys. Lett.* **109**(2), 24002 (2015)
25. K. Kamrin, Non-locality in granular flow: Phenomenology and modeling approaches. *Frontiers Phys.* **7**, 116 (2019)
26. D.G. Schaeffer, Instability in the evolution equations describing incompressible granular flow. *J. Diff. Equ.* **66**(1), 19–50 (1987)
27. Q. Zhang, K. Kamrin, Microscopic description of the granular fluidity field in nonlocal flow modeling. *Phys. Rev. Lett.* **118**(5), 058001 (2017)
28. Z. Tang, T.A. Brzinski, M. Shearer, K.E. Daniels, Non-local rheology of dense granular flow in annular shear experiments. *Soft Matter* **14**(16), 3040–3048 (2018)
29. J. Gaume, G. Chambon, M. Naaim, Microscopic origin of nonlocal rheology in dense granular materials. *Phys. Rev. Lett.* **125**(18), 188001 (2020)
30. J.A. Robinson, D.J. Holland, L. Fullard, Examination of the microscopic definition for granular fluidity. *Phys. Rev. Fluids* **6**(4), 044302 (2021)
31. F. Fazelpour, Z. Tang, K.E. Daniels, The effect of grain shape and material on the nonlocal rheology of dense granular flows. *Soft Matter* **18**(7), 1435–1442 (2022)
32. D.A. Clarke, J. Poata, P. Galvosas, D.J. Holland, Investigation of nonlocal granular fluidity models using nuclear magnetic resonance. *Phys. Fluids* **36**(5) (2024)
33. R.N. Poon, A.L. Thomas, N.M. Vriend, Microscopic origin of granular fluidity: an experimental investigation. *Phys. Rev. E* **108**(6), 064902 (2023)
34. K.-L. Lee, T.-Y. Yeh, Mesoscale avalanche size underpins the rheology of granular yielding. *Proc. Nat. Acad. Sci.* **122**(40), 2516426122 (2025)
35. S. Kim, K. Kamrin, Power-law scaling in granular rheology across flow geometries. *Phys. Rev. Lett.* **125**(8), 088002 (2020)
36. B.P. Bhowmik, C. Ness, Unifying homogeneous and inhomogeneous rheology of dense suspensions. *J. Rheology* **69**(4), 423–433 (2025)
37. P. Shekari, B. Marks, P. Rognon, Size of heterogeneous deformations in sheared granular flows. *Phys. Rev. Fluids* **8**(12), 124301 (2023)
38. T. Barker, D.G. Schaeffer, P. Bohórquez, J.M.N.T. Gray, Well-posed and ill-posed behaviour of the  $\mu(I)$ -rheology for granular flows. *J. Fluid Mech.* **779**, 794–818 (2015)
39. J.D. Goddard, Continuum modeling of granular media. *App. Mech. Rev.* **66**(5), 050801 (2014)
40. D.L. Henann, K. Kamrin, Continuum thermomechanics of the nonlocal granular rheology. *Int. J. Plast* **60**, 145–162 (2014)
41. T. Barker, J.M.N.T. Gray, Partial regularisation of the incompressible  $\mu(I)$ -rheology for granular flow. *J. Fluid Mech.* **828**, 5–32 (2017)
42. A. Franci, M. Cremonesi, 3d regularized  $\mu(I)$ -rheology for granular flows simulation. *J. Comp. Phys.* **378**, 257–277 (2019)
43. T. Barker, D.G. Schaeffer, M. Shearer, J.M.N.T. Gray, Well-posed continuum equations for granular flow with compressibility and  $\mu(I)$ -rheology. *Proc. R. Soc. A* **473**(2201), 20160846 (2017)
44. J. Heyman, R. Delannay, H. Tabuteau, A. Valance, Compressibility regularizes the  $\mu(i)$  rheology for dense granular flows. *J. Fluid Mech.* **830**, 553–568 (2017)
45. D. Schaeffer, T. Barker, D. Tsuji, P. Gremaud, M. Shearer, J. Gray, Constitutive relations for compressible granular flow in the inertial regime. *J. Fluid Mech.* **874**, 926–951 (2019)
46. S. Li, D.L. Henann, Material stability and instability in non-local continuum models for dense granular materials. *J. Fluid Mech.* **871**, 799–830 (2019)
47. D. Liu, D.L. Henann, Non-local continuum modelling of steady, dense granular heap flows. *J. Fluid Mech.* **831**, 212–227 (2017)
48. C.-C. Lin, F.-L. Yang, Continuum simulation for regularized non-local  $\mu(i)$  model of dense granular flows. *J. Comp. Phys.* **420**, 109708 (2020)
49. C.-C. Lin, F.-L. Yang, Continuum simulation of non-local effects in a granular silo discharge flow using a regularized  $\mu(i)$  rheology model. *Phys. Fluids* **33**(9) (2021)
50. T. Xu, Y.-C. Jin, Two-dimensional continuum modelling granular column collapse by non-local peridynamics in a mesh-free method with rheology. *J. Fluid Mech.* **917**, 51 (2021)

51. A. Haeri, K. Skonieczny, Three-dimensional granular flow continuum modeling via material point method with hyperelastic nonlocal granular fluidity. *Comp. Meth. App. Mech. Eng.* **394**, 114904 (2022)
52. S. Mowlavi, K. Kamrin, Interplay between hysteresis and nonlocality during onset and arrest of flow in granular materials. *Soft Matter* **17**(31), 7359–7375 (2021)
53. S. Kim, K. Kamrin, A second-order non-local model for granular flows. *Frontiers Phys.* **11**, 1092233 (2023)
54. D. Faroux, K. Washino, T. Tsuji, T. Tanaka, Granular fluidity in cohesive split-bottom granular flows. *Phys. Rev. Fluids* **7**(8), 084306 (2022)
55. S. Mandal, M. Nicolas, O. Pouliquen, Rheology of cohesive granular media: shear banding, hysteresis, and non-local effects. *Phys. Rev. X* **11**(2), 021017 (2021)

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.